

Rényi and Tsallis formulations of separability conditions in finite dimensions

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Separability conditions for a bipartite quantum systems of finite-dimensional subsystems are formulated in terms of Rényi and Tsallis entropies. Since entangled states underlie many protocols of quantum information, methods of entanglement detection are of great importance. Entropic functions provide a good quantitative measure in studying various questions including uncertainty in quantum measurement. Entropic uncertainty relations often give rise to entanglement criteria. Our approach is based on the convolution operation applied to discrete probability distributions. Measurements on a total system are constructed of local ones according to the convolution scheme. Separability conditions are derived on the base of uncertainty relations of the Maassen–Uffink type as well as majorization relations. On each of subsystems, we use a pair of sets of subnormalized vectors that form rank-one POVMs. We also obtain entropic separability conditions for local measurements with a special structure, such as mutually unbiased bases and symmetric informationally complete measurements. Their extensions known as mutually unbiased measurements and general SIC-POVMs are addressed as well.

Keywords: entropic uncertainty principle, convolution, majorization, separable states

I. INTRODUCTION

Quantum entanglement stands amongst the fundamental properties that differentiate the quantum world from the classical. This quantum-mechanical feature was concerned by founders in the Schrödinger “cat paradox” paper [1] and in the Einstein–Podolsky–Rosen paper [2]. Entanglement is central to all questions of the emerging technologies of quantum information science. Hence, various aspects of quantum entanglement are currently the subject of active research (see, e.g., the review [3] and references therein). Due to progress in quantum information processing, both the detection and quantification of entanglement are very important. In the case of discrete variables, the positive partial transpose (PPT) criterion [4] and the reduction criterion [5] are very powerful. On the other hand, no universal criteria are known even for discrete variables. Say, the PPT criterion is necessary and sufficient for 2×2 and 2×3 systems, but ceases to be so in higher dimensions [6]. Separability conditions can be derived from uncertainty relations of various forms [7–13].

Since the Heisenberg uncertainty appeared [14], many formulations and scenarios were studied to understand uncertainty of complementary observables [15, 16]. One of approaches to quantifying uncertainty in quantum measurements is based on the use of entropies [17–19]. The first entropic uncertainty relation for position and momentum was derived in [20] and later improved in [21, 22]. Entropic uncertainty relations are currently the subject of active research [23–25]. Entropic bounds cannot distinguish the uncertainty inherent in obtaining a particular combination of the outcomes [26]. Fine-grained uncertainty relations were studied for several scenarios [27, 28]. Majorization approach provides an alternative way to express the uncertainty principle in terms of probabilities *per se* [29]. Majorization relations in finite dimensions were formulated in [30–32]. The author of [33] derived coarse-grained counterparts of discrete uncertainty relations based on the concept of majorization. Majorization uncertainty relations for quantum operations were examined in [34]. Majorization-based entropic bounds are sometimes better than bounds of the Maassen–Uffink type [30, 32].

The aim of the present work is to derive separability conditions on the base of local entropic bounds of several types. In particular, our approach allows to compare majorization uncertainty relations with relations of the Maassen–Uffink type in the context of their application in entanglement detection. For construction of measurement operators on the total system, we will use a unifying scheme proposed in [13]. It will be shown that this scheme can be extended to the case of POVMs. Our scheme of constructing measurement on the total system emphasizes the role of convolution operation. The paper is organized as follows. In Sect. II, we review required material of matrix analysis and several facts concerning the convolution operation with discrete indices. In Sect. III, some known forms of entropic uncertainty relations are recalled. Separability conditions in terms of Rényi and Tsallis entropies are formulated in Sect. IV. Examples of application of derived criteria are discussed in Sect. V.

II. PRELIMINARIES

In this section, we recall the required material and describe the notation. For two integers $m, n \geq 1$ the symbol $\mathbb{M}_{m \times n}(\mathbb{C})$ denotes the space of all $m \times n$ complex matrices [35]. In the case of square matrices, when $m = n$, we write

$\mathbb{M}_n(\mathbb{C})$. Each matrix $\mathbf{M} \in \mathbb{M}_{m \times n}(\mathbb{C})$ can be represented in terms of the singular value decomposition [36],

$$\mathbf{M} = \mathbf{U}_m \mathbf{\Sigma} \mathbf{U}_n^\dagger, \quad (2.1)$$

where $\mathbf{U}_m \in \mathbb{M}_m(\mathbb{C})$ and $\mathbf{U}_n \in \mathbb{M}_n(\mathbb{C})$ are unitary. The $m \times n$ matrix $\mathbf{\Sigma} = [[\varsigma_{ij}]]$ has real entries with $\varsigma_{ij} = 0$ for all $i \neq j$. If the given matrix \mathbf{M} has rank r , then diagonal entries of $\mathbf{\Sigma}$ can be chosen so that

$$\varsigma_{11} \geq \dots \geq \varsigma_{rr} > 0 = \varsigma_{r+1,r+1} = \dots = \varsigma_{\ell\ell}, \quad (2.2)$$

where $\ell = \min\{m, n\}$. In terms of the singular values, the spectral norm is expressed as

$$\|\mathbf{M}\|_\infty = \max\{\varsigma_{jj}(\mathbf{M}) : 1 \leq j \leq \ell\}. \quad (2.3)$$

The space of linear operators on d -dimensional Hilbert space \mathcal{H} will be denoted as $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}_{s.a.}(\mathcal{H})$ and $\mathcal{L}_+(\mathcal{H})$, we mean the real space of Hermitian operators and the set of positive operators, respectively. The state of a d -level quantum system is described by density matrix $\boldsymbol{\rho} \in \mathcal{L}_+(\mathcal{H})$ normalized as $\text{Tr}(\boldsymbol{\rho}) = 1$. With respect to the prescribed basis, vectors of \mathcal{H} are represented by elements of $\mathbb{M}_{d \times 1}(\mathbb{C})$, whereas operators of $\mathcal{L}(\mathcal{H})$ are represented by elements of $\mathbb{M}_d(\mathbb{C})$.

To formulate schemes for detecting entanglement, we have to build total-system measurements of local ones. We will deal with local measurements that are described by sets of subnormalized vectors. In the simplest case one chooses some orthonormal basis $\mathcal{E} = \{|e_i\rangle\}$, $i = 0, \dots, d-1$. For the given pre-measurement state $\boldsymbol{\rho}$, i -th outcome appears with the probability $p_i(\mathcal{E}|\boldsymbol{\rho}) = \langle e_i | \boldsymbol{\rho} | e_i \rangle$. Generalized quantum measurement are typically described in terms of POVMs. Let $\mathcal{N} = \{\mathbf{N}_i\}$ be a set of elements of $\mathcal{L}_+(\mathcal{H})$, and let the completeness relation holds:

$$\sum_{i=0}^{D-1} \mathbf{N}_i = \mathbb{1}_d. \quad (2.4)$$

Such operators form a positive operator-valued measure (POVM). The probability of i -th outcome is expressed as

$$p_i(\mathcal{N}|\boldsymbol{\rho}) = \text{Tr}(\mathbf{N}_i \boldsymbol{\rho}). \quad (2.5)$$

In opposite to von Neumann measurements, the number D of different outcomes can exceed the dimensionality of the Hilbert space.

Let $p = \{p_i\}$ be a probability distribution. For $0 < \alpha \neq 1$, the Rényi α -entropy is defined as

$$R_\alpha(p) := \frac{1}{1-\alpha} \ln \left(\sum_i p_i^\alpha \right). \quad (2.6)$$

This entropy is a non-increasing function of α [37]. In the limit $\alpha \rightarrow 1$, we obtain the usual Shannon entropy

$$H_1(p) := - \sum_i p_i \ln p_i. \quad (2.7)$$

The limit $\alpha \rightarrow 0^+$ leads to the so-called max-entropy equal to the logarithm of the number of non-zero probabilities. We will not use the max-entropy in the following. For $\alpha \in (0, 1)$, the right-hand side of (2.6) is certainly concave [38]. Convexity properties of Rényi's entropies with orders $\alpha > 1$ depend on dimensionality of probabilistic vectors [39, 40]. It was shown in [40] that the binary Rényi entropy is concave for $0 < \alpha \leq 2$. We also note that the Rényi entropy is Schur-concave.

Tsallis entropies form another important extension of the Shannon entropy. For $0 < \alpha \neq 1$, the Tsallis α -entropy is defined as

$$H_\alpha(p) := \frac{1}{1-\alpha} \left(\sum_i p_i^\alpha - 1 \right). \quad (2.8)$$

For $\alpha = 1$, this entropy also reduces to (2.7). The right-hand side of (2.8) is a concave function of probability distribution for all $0 < \alpha \neq 1$. It is Schur-concave as well. Other properties of Rényi and Tsallis entropies with quantum applications are discussed in [39].

It will be convenient to define the notion of norm-like functionals. For arbitrary $\alpha > 0$, we define

$$\|p\|_\alpha := \left(\sum_i p_i^\alpha \right)^{1/\alpha}. \quad (2.9)$$

It is a legitimate norm only for $\alpha \geq 1$. For $0 < \alpha \neq 1$, we have

$$R_\alpha(p) = \frac{\alpha}{1-\alpha} \ln \|p\|_\alpha. \quad (2.10)$$

In a similar manner, the entropy (2.8) can be expressed via (2.9). By $R_\alpha(\mathcal{N}|\boldsymbol{\rho})$ and $H_\alpha(\mathcal{N}|\boldsymbol{\rho})$, we will mean the entropies obtained by substituting the probabilities (2.5) into (2.6) and (2.8), respectively.

We now consider two functions $g = \{g_i\}$ and $h = \{h_i\}$ of the discrete variable i that runs D points. Using POVMs, dimensionality of probabilistic vectors may exceed d . Then the convolution of g and h is introduced as

$$(g * h)_k := \sum_{i=0}^{D-1} g_i h_{k \ominus i}, \quad (2.11)$$

where the sign “ \ominus ” denotes the subtraction in \mathbb{Z}/D . The following statement should be formulated explicitly.

Proposition 1 *Let $g = \{g_i\}$ and $h = \{h_i\}$ be positive-valued functions of $i \in \{0, 1, \dots, D-1\}$, and let*

$$\sum_i h_i = 1. \quad (2.12)$$

For $\alpha > 1 > \beta > 0$, we then have

$$\|g * h\|_\alpha \leq \|g\|_\alpha, \quad (2.13)$$

$$\|g * h\|_\beta \geq \|g\|_\beta. \quad (2.14)$$

Proof. For $\alpha > 1$, the function $\xi \mapsto \xi^\alpha$ has positive second derivative. Due to Jensen’s inequality, for each k we obtain

$$[(g * h)_k]^\alpha \leq \sum_i h_{k \ominus i} g_i^\alpha. \quad (2.15)$$

Summing this with respect to k gives $\|g * h\|_\alpha^\alpha \leq \|g\|_\alpha^\alpha$ due to (2.12). This completes the proof of (2.13). For $0 < \beta < 1$, the function $\xi \mapsto \xi^\beta$ has negative second derivative. Rewriting the above inequalities in opposite direction, we get the claim (2.14). ■

We will also use the notion of majorization. Let us treat real-valued functions $g = \{g_i\}$ and $h = \{h_i\}$ of $i \in \{0, 1, \dots, D-1\}$ as D -dimensional vectors. The formula $g \prec h$ implies that, for all $0 \leq k \leq D-1$,

$$\sum_{i=0}^k g_i^\downarrow \leq \sum_{i=0}^k h_i^\downarrow, \quad \sum_{i=0}^{D-1} g_i = \sum_{i=0}^{D-1} h_i. \quad (2.16)$$

Here, the arrows down imply that the values should be put in the decreasing order. We are now able to make a simple but important observation.

Proposition 2 *Let p and q be two probability distributions supported on the same finite set; then*

$$p * q \prec p, \quad p * q \prec q. \quad (2.17)$$

Proof. It will be sufficient to prove only one of the two relations (2.17). Let us represent each probability distribution as a column with D entries. The following result is well known (see, e.g., theorem II.1.10 of [41]). The relation $p * q \prec p$ holds if and only if

$$p * q = \mathbb{T}p \quad (2.18)$$

for some doubly stochastic matrix \mathbb{T} . Recall that a square matrix is called doubly stochastic, when its entries are positive and the sum of entries is equal to 1 in each row and in each column. In the case considered, the formula (2.18) directly follows from the definition of convolution: i -th row of \mathbb{T} reads $q_{i \ominus j}$, where j runs from 0 to $D-1$. Hence, each row of \mathbb{T} is obtained by cyclic shift of the above row by one step to the right. Thus, each probability q_j appears exactly one time in any row and in any column. Actually, the matrix \mathbb{T} is doubly stochastic. ■

In principle, the statement of Proposition 2 is related to lemma 1 of [7]. Here, we prefer to give another formulation with emphasizing the role of convolution. In addition, our scheme is rather formulated in terms of measurement operators including the case of POVMs. As lemma 1 of [7] deals with observables, it does not seem to be applicable immediately in our settings.

The task of determination of unknown state is closely related to choices of used measurement. In entanglement detection, specially designed measurements are typically used. When we have many copies of the same quantum state, we will tend to measure the state in several mutually complementary bases. For example, the state of a spin-1/2 system is considered to be estimated with measurements of the three orthogonal components of spin. In effect, the three MUBs are used here.

Let $\mathcal{E} = \{|e_i\rangle\}$ and $\mathcal{E}' = \{|e'_j\rangle\}$ be two orthonormal bases in a d -dimensional Hilbert space \mathcal{H} . They are said to be mutually unbiased if and only if for all i and j ,

$$|\langle e_i | e'_j \rangle| = \frac{1}{\sqrt{d}}. \quad (2.19)$$

Several orthonormal bases form a set of mutually unbiased bases (MUBs), when each two from them are mutually unbiased. Mutually unbiased bases have found use in many questions of quantum information theory (see [42] and references therein). When the eigenbases of two observables are mutually unbiased, the measurement of one observable reveals no information about possible outcomes of the measurement of other. When d is a prime power, we certainly have a construction of $d + 1$ MUBs [42]. It is based on properties of prime powers and corresponding finite field [43, 44]. In general, however, the maximal number of MUBs in d dimensions is still an open question [42].

Concerning the task of determination of unknown state, the following fact is essential. There exist measurements such that each of them uniquely determine every possible state by the measurement statistics that it alone generates. Measurements with this property are called informationally complete [45]. In addition, symmetric informationally complete measurements enjoy a symmetric structure in their elements. In d -dimensional Hilbert space, we consider a set of d^2 rank-one operators of the form

$$|f_i\rangle\langle f_i| = \frac{1}{d} |\phi_i\rangle\langle\phi_i|. \quad (2.20)$$

If the normalized vectors $|\phi_j\rangle$ all satisfy the condition

$$|\langle\phi_i|\phi_j\rangle|^2 = \frac{1}{d+1} \quad (i \neq j), \quad (2.21)$$

the set of operators (2.20) is a symmetric informationally complete POVM (SIC-POVM) [46]. It was conjectured that SIC-POVMs exist in all dimensions [47]. The existence of SIC-POVMs has been shown analytically or numerically for all dimensions up to 67 [48]. Interesting connections between MUBs and SIC-POVMs are discussed in [49, 50].

Since basic constructions of MUBs are related to prime power d , one can try to get an appropriate modification. The authors of [51] proposed the concept of mutually unbiased measurements (MUMs). Using weaker requirements, a complete set of $d + 1$ MUMs exists for all d . Let us consider two POVM measurements $\mathcal{N} = \{\mathbf{N}_i\}$ and $\mathcal{N}' = \{\mathbf{N}'_j\}$. Each of them contains d elements such that

$$\text{Tr}(\mathbf{N}_i) = \text{Tr}(\mathbf{N}'_j) = 1, \quad (2.22)$$

$$\text{Tr}(\mathbf{N}_i \mathbf{N}'_j) = \frac{1}{d}. \quad (2.23)$$

The POVM elements are all of trace one, but generally not of rank one. The formula (2.23) is used instead of the squared formula (2.19). Two different elements of the same POVM \mathcal{N} satisfy

$$\text{Tr}(\mathbf{N}_i \mathbf{N}_j) = \delta_{ij} \varkappa + (1 - \delta_{ij}) \frac{1 - \varkappa}{d - 1}, \quad (2.24)$$

where \varkappa is the efficiency parameter [51]. The same condition is imposed on the elements of \mathcal{N}' . By \varkappa , we characterize how close the POVM elements are to rank-one projectors [51]. In general, one satisfies [51]

$$\frac{1}{d} < \varkappa \leq 1. \quad (2.25)$$

For $\varkappa = 1/d$ we have the trivial case, in which $\mathbf{N}_i = \mathbf{1}/d$ for all i . The value $\varkappa = 1$, if possible, gives the standard case of mutually unbiased bases. In principle, we can only say that the maximal efficiency can be reached for prime power d . More precise bounds on \varkappa depend on an explicit construction of POVM elements [51].

Similar ideas can be used in building a generalization of SIC-POVMs. For all finite d , a common construction has been given [52]. Consider a POVM with d^2 elements \mathbf{N}_i , which satisfy the following two conditions. First, for all $i = 0, \dots, d^2 - 1$ we have

$$\text{Tr}(\mathbf{N}_i \mathbf{N}_i) = a. \quad (2.26)$$

Second, the pairwise inner products are all symmetrical, namely

$$\text{Tr}(\mathbf{N}_i \mathbf{N}_j) = b \quad (i \neq j). \quad (2.27)$$

Then the operators \mathbf{N}_i form a general SIC-POVM. Combining the conditions (2.26) and (2.27) with the completeness relation finally gives [52]

$$b = \frac{1 - ad}{d(d^2 - 1)}. \quad (2.28)$$

We also get $\text{Tr}(\mathbf{N}_i) = 1/d$ for all i . Thus, the value a is the only parameter that characterizes the type of a general SIC-POVM. In general, this parameter is restricted as [52]

$$\frac{1}{d^3} < a \leq \frac{1}{d^2}. \quad (2.29)$$

The value $a = 1/d^3$ leads to the case $\mathbf{N}_i = \mathbf{1}_d/d^2$, which does not give an informationally complete POVM. The value $a = 1/d^2$ is achieved, when the POVM elements are all rank-one [52]. The latter is actually the case of usual SIC-POVMs, when POVM elements are represented in terms of the corresponding unit vectors as (2.20). Even if usual SIC-POVMs exist in all dimensions, they are rather hard to construct. General SIC-POVMs have a similar structure that makes them appropriate in determining an informational content of a quantum state.

III. SOME FORMS OF UNCERTAINTY RELATIONS

In this section, we briefly recall some of existing formulations of the uncertainty principle. To formulate separability conditions, we will use two or more orthonormal bases in d -dimensional Hilbert space \mathcal{H} and, further, two or more POVMs. First, we will address uncertainty relations of the Maassen–Uffink type [19]. Second, majorization uncertainty relations of the papers [30, 32] will be applied. Uncertainty relations for measurements with a special structure should also be considered.

The used orthonormal bases are denoted by $\mathcal{E} = \{|e_i\rangle\}$ and $\mathcal{E}' = \{|e'_j\rangle\}$ with $i, j = 0, \dots, d-1$. If the pre-measurement state is described by normalized density matrix $\rho \in \mathcal{L}_+(\mathcal{H})$, then elements of the generated probability distributions are expressed as

$$p_i = p_i(\mathcal{E}|\rho), \quad q_j = p_j(\mathcal{E}'|\rho). \quad (3.1)$$

Entropic uncertainty relations of the Maassen–Uffink type were obtained in [19]. This issue was inspired by a conjecture of Kraus [18]. To the orthonormal bases $\mathcal{E} = \{|e_i\rangle\}$ and $\mathcal{E}' = \{|e'_j\rangle\}$, we assign

$$\eta(\mathcal{E}, \mathcal{E}') := \max \left\{ |\langle e_i | e'_j \rangle| : 0 \leq i, j \leq d-1 \right\}. \quad (3.2)$$

Due to Riesz's theorem [66], we have the following relation between norm-like functionals of probability distribution,

$$\|p\|_\alpha \leq \eta^{2(1-\beta)/\beta} \|q\|_\beta, \quad (3.3)$$

$$\|q\|_\alpha \leq \eta^{2(1-\beta)/\beta} \|p\|_\beta, \quad (3.4)$$

where $1/\alpha + 1/\beta = 2$ and $\alpha > 1 > \beta$. Hence, various uncertainty relations in terms of generalized entropies can be derived. For some reasons, however, we will begin derivation of separability conditions just with (3.3) and (3.4). Here, the basic point is that we generally have concavity of the Rényi α -entropy only for $0 < \alpha \leq 1$.

The formulas (3.3) and (3.4) can be generalized to POVM measurements, when non-orthogonal resolutions of the identity are used [53]. Here, we restrict a consideration to especially important case of rank-one POVMs. Let $\mathcal{F} = \{|f_i\rangle\}$ and $\mathcal{F}' = \{|f'_j\rangle\}$ be two sets of D subnormalized vectors such that

$$\sum_{i=0}^{D-1} |f_i\rangle\langle f_i| = \mathbf{1}_d, \quad \sum_{j=0}^{D-1} |f'_j\rangle\langle f'_j| = \mathbf{1}_d. \quad (3.5)$$

The number D of subnormalized vectors exceeds dimensionality of the Hilbert space. As was noted in [54], the Maassen–Uffink uncertainty relation is immediately generalized to the case of such measurements. We are rather

interested in extending just (3.3) and (3.4). The corresponding derivation from Riesz's theorem was given in [55]. Replacing (3.2) with

$$\eta(\mathcal{F}, \mathcal{F}') := \max \left\{ |\langle f_i | f'_j \rangle| : 0 \leq i, j \leq D-1 \right\}, \quad (3.6)$$

the relations (3.3) and (3.4) still hold under the conditions $1/\alpha + 1/\beta = 2$ and $\alpha > 1 > \beta$.

We now recall the majorization approach to uncertainty relations in finite dimensions. Applications of this approach beyond this case are discussed in [29, 33]. Suppose p and q denote two probability vectors generated by two quantum measurements in the same prepared state. The basic idea of is to majorize some binary combination of p and q by a third vector with bounding elements. To do so, the authors of [30, 32] inspected norms of submatrices of a certain unitary matrix.

To the orthonormal bases $\mathcal{E} = \{|e_i\rangle\}$ and $\mathcal{E}' = \{|e'_j\rangle\}$, one assigns the unitary $d \times d$ matrix $V(\mathcal{E}, \mathcal{E}')$ with entries $v_{ij} = \langle e_i | e'_j \rangle$. By $SUB(V, k)$, we mean the set of all its submatrices of class k defined by

$$SUB(V, k) := \{M \in \mathbb{M}_{r \times r'}(\mathbb{C}) : r + r' = k + 1, M \text{ is a submatrix of } V\}. \quad (3.7)$$

The positive integer k runs all the values allowed by the condition $r + r' = k + 1$. The majorization relations of [30, 32] are expressed in terms of quantities

$$s_k := \max \{ \|M\|_\infty : M \in SUB(V, k) \}. \quad (3.8)$$

It will be convenient to numerate these quantities by integers starting with 1. Due to completeness and orthonormality of each bases, one has $s_d = 1$ and, therefore, $s_k = 1$ for all $d \leq k \leq 2d - 1$.

The authors of [32] proved the majorization relation

$$p \oplus q \prec \{1\} \oplus w, \quad (3.9)$$

$$w = (s_1, s_2 - s_1, \dots, s_d - s_{d-1}). \quad (3.10)$$

This majorizing vector is completed by $s_d - s_{d-1}$, since $s_k = 1$ for $d \leq k \leq 2d - 1$ and further differences are all zero. The following entropic bounds follow from (3.9). For $0 < \alpha \leq 1$, it holds that

$$R_\alpha(p) + R_\alpha(q) \geq R_\alpha(w). \quad (3.11)$$

For $\alpha > 1$, the sum of two Rényi entropies obeys another inequality [32]

$$R_\alpha(p) + R_\alpha(q) \geq \frac{2}{1 - \alpha} \ln \left(\frac{1 + \|w\|_\alpha^\alpha}{2} \right). \quad (3.12)$$

Majorization relations of the tensor-product type were first considered in [30, 31]. The authors of [30] showed that

$$p \otimes q \prec w', \quad (3.13)$$

where the majorizing vector

$$w' = (t_1, t_2 - t_1, \dots, t_d - t_{d-1}), \quad t_k = \frac{(1 + s_k)^2}{4}. \quad (3.14)$$

The majorization relation (3.13) implies that, for $\alpha > 0$ [30, 31],

$$R_\alpha(p) + R_\alpha(q) \geq R_\alpha(w'). \quad (3.15)$$

It follows from that the Rényi entropy is Schur-concave. For $0 < \alpha \leq 1$, we will prefer (3.11), since $\omega \prec \omega'$ and $R_\alpha(\omega) \geq R_\alpha(\omega')$ [32]. Nevertheless, the relation (3.15) may be useful for $\alpha > 1$. The sum of two Tsallis α -entropies is bounded from below similarly to (3.11). For any $\alpha > 0$ we have [32]

$$H_\alpha(p) + H_\alpha(q) \geq H_\alpha(w). \quad (3.16)$$

As we plan to deal with rank-one POVMs, the majorization approach should be reformulated appropriately. In principle, a general way of extension was considered in [34]. However, that paper focus on quantum operations described in terms of Kraus operators. The case of rank-one POVMs is so important that we prefer to give an explicit derivation. In this case, the majorization approach is based on the following statement.

Proposition 3 Let each of sets $\mathcal{F} = \{|f_i\rangle\}$ and $\mathcal{F}' = \{|f'_j\rangle\}$ contain D subnormalized vectors that form rank-one POVM in d -dimensional space \mathcal{H} . Let \mathcal{I} and \mathcal{J} be two subsets of the set $\{0, \dots, D-1\}$. For arbitrary density matrix ρ , we have

$$\sum_{i \in \mathcal{I}} p_i(\mathcal{F}|\rho) + \sum_{j \in \mathcal{J}} p_j(\mathcal{F}'|\rho) \leq 1 + \|\mathbf{C}_{\mathcal{I}}\mathbf{C}_{\mathcal{J}}^\dagger\|_\infty. \quad (3.17)$$

Here, the $|\mathcal{I}| \times d$ matrix $\mathbf{C}_{\mathcal{I}}$ is formed by rows $\langle f_i|$ with $i \in \mathcal{I}$, and the $|\mathcal{J}| \times d$ matrix $\mathbf{C}_{\mathcal{J}}$ is formed by rows $\langle f'_j|$ with $j \in \mathcal{J}$.

Proof. For definiteness, we write $\mathcal{I} = \{i_1, \dots, i_m\}$ and $\mathcal{J} = \{j_1, \dots, j_n\}$, whence

$$\mathbf{C}_{\mathcal{I}} = \begin{pmatrix} \langle f_{i_1}| \\ \vdots \\ \langle f_{i_m}| \end{pmatrix}, \quad \mathbf{C}_{\mathcal{J}} = \begin{pmatrix} \langle f'_{j_1}| \\ \vdots \\ \langle f'_{j_n}| \end{pmatrix}. \quad (3.18)$$

It will be sufficient to prove the claim (3.17) for pure states. Its validity for mixed states follows by the spectral decomposition. Keeping in mind matrix relations of the form

$$\mathbf{C}_{\mathcal{I}}^\dagger \mathbf{C}_{\mathcal{I}} = \sum_{i \in \mathcal{I}} |f_i\rangle\langle f_i|, \quad (3.19)$$

we have

$$\sum_{i \in \mathcal{I}} p_i(\mathcal{F}|\psi) + \sum_{j \in \mathcal{J}} p_j(\mathcal{F}'|\psi) = \langle \psi | \mathbf{G}^\dagger \mathbf{G} | \psi \rangle, \quad \mathbf{G} = \begin{pmatrix} \mathbf{C}_{\mathcal{I}} \\ \mathbf{C}_{\mathcal{J}} \end{pmatrix}. \quad (3.20)$$

Due to properties of the spectral norm, we obtain

$$\langle \psi | \mathbf{G}^\dagger \mathbf{G} | \psi \rangle \leq \|\mathbf{G}^\dagger \mathbf{G}\|_\infty = \|\mathbf{G} \mathbf{G}^\dagger\|_\infty \leq \max\{\|\mathbf{C}_{\mathcal{I}}\|_\infty^2, \|\mathbf{C}_{\mathcal{J}}\|_\infty^2\} + \|\mathbf{C}_{\mathcal{I}}\mathbf{C}_{\mathcal{J}}^\dagger\|_\infty. \quad (3.21)$$

The justification of (3.21) is very similar to the proof of proposition 2 of [34]. The definition of $\mathbf{C}_{\mathcal{I}}$ and $\mathbf{C}_{\mathcal{J}}$ is the only distinction. Let us put the complete $D \times d$ matrices such that $\mathbf{C}_{\mathcal{F}}$ is formed by all the rows $\langle f_i|$, and $\mathbf{C}_{\mathcal{F}'}$ is formed by all the rows $\langle f'_j|$. By submultiplicativity of the spectral norm, one gets

$$\|\mathbf{C}_{\mathcal{I}}\|_\infty \leq \|\mathbf{C}_{\mathcal{F}}\|_\infty, \quad \|\mathbf{C}_{\mathcal{J}}\|_\infty \leq \|\mathbf{C}_{\mathcal{F}'}\|_\infty. \quad (3.22)$$

It follows from (3.24) that

$$\mathbf{C}_{\mathcal{F}}^\dagger \mathbf{C}_{\mathcal{F}} = \mathbf{C}_{\mathcal{F}'}^\dagger \mathbf{C}_{\mathcal{F}'} = \mathbb{1}_d, \quad (3.23)$$

whence $\|\mathbf{C}_{\mathcal{F}}\|_\infty = \|\mathbf{C}_{\mathcal{F}'}\|_\infty = 1$. Combining the latter with (3.21) and (3.22) completes the proof. ■

Thus, lemma 1 of [30] is extended from two orthonormal bases to two rank-one POVMs. For orthonormal bases, we will have

$$\mathbf{C}_{\mathcal{I}}^\dagger \mathbf{C}_{\mathcal{I}} = \mathbb{1}_m, \quad \mathbf{C}_{\mathcal{J}}^\dagger \mathbf{C}_{\mathcal{J}} = \mathbb{1}_n, \quad \|\mathbf{C}_{\mathcal{I}}\|_\infty = \|\mathbf{C}_{\mathcal{J}}\|_\infty = 1. \quad (3.24)$$

For rank-one POVMs, both the norms $\|\mathbf{C}_{\mathcal{I}}\|_\infty$ and $\|\mathbf{C}_{\mathcal{J}}\|_\infty$ do not exceed 1 that is sufficient for our aims.

Using (3.17), we can now extend (3.11), (3.12), (3.15), and (3.16). Denoting $p = p(\mathcal{F}|\rho)$ and $q = p(\mathcal{F}'|\rho)$, these uncertainty relations are all valid with the following changes. The quantities (3.8) are now calculated with $1 \leq k \leq 2D-1$ for the $D \times D$ matrix

$$\mathbf{V}(\mathcal{F}, \mathcal{F}') = [[\langle f_i | f'_j \rangle]]. \quad (3.25)$$

Combining (3.23) with $\mathbf{V} = \mathbf{C}_{\mathcal{F}} \mathbf{C}_{\mathcal{F}'}^\dagger$ leads to $\mathbf{V}^\dagger \mathbf{V} = \mathbf{C}_{\mathcal{F}} \mathbf{C}_{\mathcal{F}}^\dagger$ and

$$\|\mathbf{V}^\dagger \mathbf{V}\|_\infty = \|\mathbf{C}_{\mathcal{F}} \mathbf{C}_{\mathcal{F}}^\dagger\|_\infty = 1, \quad (3.26)$$

whence $\|\mathbf{V}\|_\infty = 1$. Thus, we certainly have $s_k = 1$ for $k = 2D-1$. Let D_\star denote the first index with the property $s_{D_\star} = 1$. Since vectors of the sets \mathcal{F} and \mathcal{F}' are all subnormalized, we will have

$$D \leq D_\star \leq 2D-1. \quad (3.27)$$

The vectors (3.10) and (3.14) should then include differences up to $s_{D_\star} - s_{D_\star-1}$ and $t_{D_\star} - t_{D_\star-1}$, respectively. In the following, the uncertainty relations of this section will be applied in deriving entanglement criteria. Among majorization-based relations, we will mainly use (3.11) and (3.16) with respect to the values of α , for which the corresponding entropy is certainly concave.

We have discussed entropic uncertainty relations applicable to a pair of arbitrary rank-one POVMs. It is often expedient to check entanglement with specially designed measurements. For example, mutually unbiased measurements are considered to be capable for such purposes. another interesting way is connected with symmetric informationally complete measurements. Here, we will use entropic uncertainty relations derived in [55].

Let $\{\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(K)}\}$ be a set of K MUBs in d -dimensional space \mathcal{H} . For $\alpha \in (0; 2]$, the sum of Rényi's entropies satisfies the state-independent bound

$$\frac{1}{K} \sum_{t=1}^K R_\alpha(\mathcal{E}^{(t)}|\rho) \geq \ln\left(\frac{Kd}{d+K-1}\right). \quad (3.28)$$

Note that the right-hand side of (3.28) is independent of α , whereas Rényi's entropy does not increase with growth of α . To obtain more sensitive criteria, we should take largest orders providing concavity of entropies. Therefore, we will use (3.28) with $\alpha = 1$ for arbitrary d and with $\alpha = 2$ for $d = 2$. For $\alpha \in (0; 2]$ and arbitrary state ρ on \mathcal{H} , the sum of Tsallis' entropies satisfies the state-independent bound

$$\frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{E}^{(t)}|\rho) \geq \ln_\alpha\left(\frac{Kd}{d+K-1}\right). \quad (3.29)$$

The results (3.28) and (3.29) are based on the inequality

$$\sum_{t=1}^K \sum_{i=0}^{d-1} p_i(\mathcal{E}^{(t)}|\rho)^2 \leq \text{Tr}(\rho^2) + \frac{K-1}{d} \leq 1 + \frac{K-1}{d}, \quad (3.30)$$

derived in [56]. Of course, the existence of K MUBs should be proved independently. We will study entropic formulation of entanglement criteria based on (3.30). It differs from the previous approach considered in [57]. Using (3.30), the authors of [57] put a specific correlation measure that is bounded from above for separable states.

For mutually unbiased measurements, the following extension of (3.30) takes place [58]. Let $\{\mathcal{N}^{(1)}, \dots, \mathcal{N}^{(K)}\}$ be a set of K MUMs of the efficiency \varkappa . For arbitrary ρ , we then have [58]

$$\sum_{t=1}^K \sum_{i=0}^{d-1} p_i(\mathcal{N}^{(t)}|\rho)^2 \leq \frac{1 - \varkappa + (\varkappa d - 1) \text{Tr}(\rho^2)}{d-1} + \frac{K-1}{d} \leq \varkappa + \frac{K-1}{d}. \quad (3.31)$$

For $d+1$ MUMs, the inequality (3.31) is actually saturated [58]. For pure states, this result was shown in [51] and then applied for entanglement detection in [59]. It follows from (3.31) that, for $\alpha \in (0; 2]$,

$$\frac{1}{K} \sum_{t=1}^K R_\alpha(\mathcal{N}^{(t)}|\rho) \geq \ln\left(\frac{Kd}{\varkappa_S d + K - 1}\right), \quad (3.32)$$

$$\frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{N}^{(t)}|\rho) \geq \ln_\alpha\left(\frac{Kd}{\varkappa_S d + K - 1}\right). \quad (3.33)$$

The existence of K MUMs of some efficiency was proved up to $K = d+1$ [51].

For a SIC-POVM $\mathcal{F} = \{|f_i\rangle\}$, the index of coincidence can also be calculated exactly. For the given pre-measurement state ρ , we have [55]

$$\sum_{i=0}^{d^2-1} p_i(\mathcal{F}|\rho)^2 = \frac{\text{Tr}(\rho^2) + 1}{d(d+1)} \leq \frac{2}{d(d+1)}. \quad (3.34)$$

As was briefly noticed in [55], this result allows to build a SIC-POVM scheme for entanglement detection. The following uncertainty relations were derived due to (3.34). For $\alpha \in (0; 2]$ and arbitrary density matrix ρ on \mathcal{H} , it holds that [55]

$$R_\alpha(\mathcal{F}|\rho) \geq \ln\left(\frac{d(d+1)}{2}\right), \quad (3.35)$$

$$H_\alpha(\mathcal{F}|\rho) \geq \ln_\alpha\left(\frac{d(d+1)}{2}\right). \quad (3.36)$$

Again, we will use (3.35) with $\alpha = 1$ for arbitrary d and with $\alpha = 2$ for $d = 2$.

Let general SIC-POVM $\mathcal{N} = \{\mathbf{N}_i\}$ be characterized by the parameter a in the sense of (2.26). For the given pre-measurement state $\boldsymbol{\rho}$, we have [60]

$$\sum_{i=0}^{d^2-1} p_i(\mathcal{N}|\boldsymbol{\rho})^2 = \frac{(ad^3 - 1)\text{Tr}(\boldsymbol{\rho}^2) + d(1 - ad)}{d(d^2 - 1)} \leq \frac{ad^2 + 1}{d(d + 1)}. \quad (3.37)$$

For a usual SIC-POVM, when $a = d^{-2}$, the result (3.37) is reduced to (3.34). For $\alpha \in (0; 2]$ and arbitrary density matrix $\boldsymbol{\rho}$ on \mathcal{H} , one gets [60]

$$R_\alpha(\mathcal{N}|\boldsymbol{\rho}) \geq \ln\left(\frac{d(d + 1)}{ad^2 + 1}\right), \quad (3.38)$$

$$H_\alpha(\mathcal{N}|\boldsymbol{\rho}) \geq \ln_\alpha\left(\frac{d(d + 1)}{ad^2 + 1}\right), \quad (3.39)$$

where the general SIC-POVM \mathcal{N} is characterized by the parameter a . Due to (3.37), general SIC-POVMs can be used for entanglement detection. The authors of [61] gave an appropriate form of correlation measures proposed in [57] and reformulated for usual SIC-POVMs in [55].

IV. FORMULATION OF SEPARABILITY CONDITIONS

In this section, we will obtain separability conditions on the base of uncertainty relations discussed in the previous section. Such uncertainty relations are local in the sense that they will be posed for one of subsystems. A utility of entanglement criteria based on local uncertainty relations was justified in [10]. To formulate such separability conditions, we should construct some total-system measurements of local ones. Before, we briefly recall some definitions.

Let us consider a bipartite system of d -level subsystems labeled by A and B . The tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the total Hilbert space. Any quantum state of the total system is described by a density matrix on $\mathcal{H}_{AB} \in \mathcal{L}_+(\mathcal{H}_{AB})$. Product states written as $\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B$ reveal no correlations between subsystems. A bipartite mixed state is called separable, when its density matrix can be represented as a convex combination of product states [62, 63]. Without loss of generality, each separable state will be treated as a convex combination of only pure product states. This point follows from the existence of the spectral decomposition. For more theoretical results about separable operators and states, see chapter 6 of [45].

Let us proceed to building measurements on a bipartite system of interest. It will be convenient to develop the scheme already motivated in [13]. Suppose we have two orthonormal bases, one in \mathcal{H}_A and one in \mathcal{H}_B . One may prefer to distribute “equally” rank-one projectors between elements of resulting measurements. This approach can naturally be described within the following construction. For all bases, the subscript runs integers from 0 up to $d - 1$. To each basis $\mathcal{E}_A = \{|e_{Ai}\rangle\}$ in \mathcal{H}_A , we assign the operator \mathbf{A} . The operator is taken as diagonal in that basis and represented as

$$\mathbf{A} = \text{diag}(1, \omega, \dots, \omega^{d-1}), \quad (4.1)$$

where $\omega = \exp(i2\pi/d)$ is a primitive root of the unit. To each basis $\mathcal{E}_B = \{|e_{Bj}\rangle\}$ in \mathcal{H}_B , we similarly the operator $\mathbf{B}^{(t)}$ expressed as the right-hand side of (4.1) in this basis. The spectrum of $\mathbf{A} \otimes \mathbf{B}$ also contains d powers of ω . For each label t , we now write

$$\mathbf{A} \otimes \mathbf{B} = \sum_{k=0}^{d-1} \omega^k \boldsymbol{\Lambda}_k. \quad (4.2)$$

In the right-hand side of (4.2), each projector is expressed as

$$\boldsymbol{\Lambda}_k = \sum_{i=0}^{d-1} |e_{Ai}\rangle\langle e_{Ai}| \otimes |e_{Bk \ominus i}\rangle\langle e_{Bk \ominus i}|. \quad (4.3)$$

To each pair of orthonormal bases, one per subsystem, we get an orthogonal resolution of $\mathbb{1}_{AB}$, i.e., $\mathcal{M}(\mathcal{E}_A, \mathcal{E}_B) = \{\boldsymbol{\Lambda}_k\}$. Except for $d = 2$, the operators (4.1) and (4.2) are not Hermitian. However, these operators are only auxiliary in

order to get the projection operators (4.3). We will usually use two or more measurements with operators of the form (4.3). It directly follows from (2.11) and (4.3) that

$$p(\mathcal{M}^{(t)}|\rho_A \otimes \rho_B) = p(\mathcal{E}_A^{(t)}|\rho_A) * p(\mathcal{E}_B^{(t)}|\rho_B), \quad (4.4)$$

where a label t marks measurements. The above scheme is immediately generalized to POVM measurements. For definiteness, we formulate this extension explicitly.

Definition 1 Let $\mathcal{N}_A = \{\mathbf{N}_{Ai}\}$ be a D -outcome POVM in \mathcal{H}_A , and $\mathcal{N}_B = \{\mathbf{N}_{Bj}\}$ be a D -outcome POVM in \mathcal{H}_B . We call a POVM $\mathcal{M}(\mathcal{N}_A, \mathcal{N}_B) = \{\Pi_k\}$ to be constructed according to the convolution scheme, when

$$\Pi_k := \sum_{i=0}^{D-1} \mathbf{N}_{Ai} \otimes \mathbf{N}_{Bk \ominus i}. \quad (4.5)$$

Here, the sign “ \ominus ” denotes the subtraction in \mathbb{Z}/D and $k \in \{0, 1, \dots, D-1\}$.

For each $\mathcal{M}^{(t)}$ built according to Definition 1, we have

$$p(\mathcal{M}^{(t)}|\rho_A \otimes \rho_B) = p(\mathcal{N}_A^{(t)}|\rho_A) * p(\mathcal{N}_B^{(t)}|\rho_B). \quad (4.6)$$

For product states, each resolution $\mathcal{M}^{(t)}$ generates the convolution of two distributions corresponding to local bases. This immediate corollary of our construction is important in deriving separability conditions. We will formulate them in the situation, when two or more different resolutions are used. Our first result is posed as follows.

Proposition 4 Let each of sets $\mathcal{F}_A^{(1)}$ and $\mathcal{F}_A^{(2)}$ of subnormalized vectors form rank-one POVM in \mathcal{H}_A , and let each of sets $\mathcal{F}_B^{(1)}$ and $\mathcal{F}_B^{(2)}$ of subnormalized vectors form rank-one POVM in \mathcal{H}_B . Let two POVMs $\mathcal{M}^{(t)}(\mathcal{F}_A^{(t)}, \mathcal{F}_B^{(t)})$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed from these sets according to Definition 1. If state ρ_{AB} is separable and $1/\alpha + 1/\beta = 2$, then

$$R_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + R_\beta(\mathcal{M}^{(2)}|\rho_{AB}) \geq -2 \ln \eta_S, \quad (4.7)$$

$$H_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + H_\beta(\mathcal{M}^{(2)}|\rho_{AB}) \geq \ln_\mu(\eta_S^{-2}), \quad (4.8)$$

where $S = A, B$, maximal entropic parameter $\mu = \max\{\alpha, \beta\}$, and $\eta_S = \eta(\mathcal{F}_S^{(1)}, \mathcal{F}_S^{(2)})$.

Proof. We will further assume that $\alpha > 1 > \beta$. The Shannon case $\alpha = \beta = 1$ is finally reached by taking the corresponding limit. It is sufficient to prove (4.7) and (4.8) only for one of the cases $S = A, B$. For brevity, we also denote

$$Q_{AB}^{(t)} = p(\mathcal{M}^{(t)}|\rho_{AB}), \quad p_A^{(t)} = p(\mathcal{F}_A^{(t)}|\rho_A), \quad q_B^{(t)} = p(\mathcal{F}_B^{(t)}|\rho_B), \quad (4.9)$$

where $t = 1, 2$, reduced densities $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$.

We first suppose that ρ_{AB} is a product state appeared as $\rho_A \otimes \rho_B$. Combining $Q_{AB}^{(t)} = p_A^{(t)} * q_B^{(t)}$ with (2.13) and (2.14), for $\alpha > 1 > \beta$ we have the inequality

$$\|Q_{AB}^{(1)}\|_\alpha \leq \|p_A^{(1)}\|_\alpha \leq \eta_A^{2(1-\beta)/\beta} \|p_A^{(2)}\|_\beta \leq \eta_A^{2(1-\beta)/\beta} \|Q_{AB}^{(2)}\|_\beta \quad (4.10)$$

and its “twin” with swapped $Q_{AB}^{(1)}$ and $Q_{AB}^{(2)}$. In general, we cannot assume concavity of the Rényi α -entropy for $\alpha > 1$. Hence, we should extend our results to separable states before obtaining final entropic inequalities.

Each separable state can be represented as a convex combination of product states, namely

$$\rho_{AB} = \sum_\lambda \lambda \rho_{A\lambda} \otimes \rho_{B\lambda}. \quad (4.11)$$

Here, density matrices are all normalized so that $\sum_\lambda \lambda = 1$. Due to (4.11), we obtain

$$Q_{AB}^{(t)} = \sum_\lambda \lambda Q_{AB\lambda}^{(t)}, \quad (4.12)$$

where each $Q_{AB\lambda}^{(t)}$ corresponds to the product $\rho_{A\lambda} \otimes \rho_{B\lambda}$. Following [64, 65], at this step we use the Minkowski inequality. Assuming $\alpha > 1 > \beta > 0$, this inequality results in

$$\|Q_{AB}^{(1)}\|_\alpha = \left\| \sum_\lambda \lambda Q_{AB\lambda}^{(1)} \right\|_\alpha \leq \sum_\lambda \lambda \|Q_{AB\lambda}^{(1)}\|_\alpha, \quad (4.13)$$

$$\sum_\lambda \lambda \|Q_{AB\lambda}^{(2)}\|_\beta \leq \left\| \sum_\lambda \lambda Q_{AB\lambda}^{(2)} \right\|_\beta = \|Q_{AB}^{(2)}\|_\beta. \quad (4.14)$$

For each λ , the quantities $\|Q_{AB\lambda}^{(1)}\|_\alpha$ and $\|Q_{AB\lambda}^{(2)}\|_\beta$ obey (4.10). By (4.13) and (4.14), the relation (4.10) and its “twin” written in terms of $Q_{AB}^{(t)}$ are also valid for all separable state.

To complete the proof, we shall convert (4.10) into entropic inequalities. The Rényi entropies are represented via norm-like functionals according to (2.10). To get (4.7) with $\alpha > 1 > \beta$, we take the logarithm of both the sides of (4.10) and use the link $(\alpha - 1)/\alpha = (1 - \beta)/\beta$. The inequality with swapped probability distributions is then obtained by a parallel argument. Together, these inequality are joined into (4.7) under the condition $1/\alpha + 1/\beta = 2$ solely. The case of Tsallis entropies is not so immediate. Following [53], we can examine a minimization problem under the restrictions imposed by (4.10) and its “twin”. Calculations resulting in (4.8) are very similar to the derivation given in appendix of [53]. ■

The statement of Proposition 4 provides entropic separability conditions based on local uncertainty relations of the Maassen–Uffink type. These formulas hold under the restriction $1/\alpha + 1/\beta = 2$. The latter reflects the fact that the Maassen–Uffink result is derived from Riesz’s theorem [66]. An alternative viewpoint is that such uncertainty relations follow from the monotonicity of the quantum relative entropy [67]. Then the link between α and β is due to the so-called duality of entropies [67]. The used scheme of constructing total measurements also allows to derive some separability conditions from majorization uncertainty relations. Our second result is posed as follows.

Proposition 5 *Let each of sets $\mathcal{F}_A^{(1)}$ and $\mathcal{F}_A^{(2)}$ of subnormalized vectors form rank-one POVM in \mathcal{H}_A , and let each of sets $\mathcal{F}_B^{(1)}$ and $\mathcal{F}_B^{(2)}$ of subnormalized vectors form rank-one POVM in \mathcal{H}_B . Let two POVMs $\mathcal{M}^{(t)}(\mathcal{F}_A^{(t)}, \mathcal{F}_B^{(t)})$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed from these sets according to Definition 1. For $S = A, B$, we introduce $D \times D$ matrix $\mathbf{V}_S = \mathbf{V}(\mathcal{F}_S^{(1)}, \mathcal{F}_S^{(2)})$ due to (3.25). To each of such two matrices, we assign the sequence of numbers according to (3.8) and the majorizing vector w_S , where $S = A, B$. For each separable state ρ_{AB} and $0 < \alpha \leq 1$, there holds*

$$R_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + R_\alpha(\mathcal{M}^{(2)}|\rho_{AB}) \geq R_\alpha(w_S). \quad (4.15)$$

For each separable state ρ_{AB} and $\alpha > 0$, there holds

$$H_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + H_\alpha(\mathcal{M}^{(2)}|\rho_{AB}) \geq H_\alpha(w_S). \quad (4.16)$$

Proof. In view of symmetry between subsystems, we will prove (4.15) and (4.16) only for one of the cases $S = A, B$. Combining (4.4) with (2.17) and using the notation (4.9) again, for each product state we write

$$R_\alpha(Q_{AB}^{(t)}) \geq R_\alpha(p_A^{(t)}), \quad (4.17)$$

$$H_\alpha(Q_{AB}^{(t)}) \geq H_\alpha(p_A^{(t)}). \quad (4.18)$$

It is essential here that both the α -entropies are Schur-concave. Due to the majorization-based relation (3.11) and (3.16), we respectively obtain

$$R_\alpha(Q_{AB}^{(1)}) + R_\alpha(Q_{AB}^{(2)}) \geq R_\alpha(p_A^{(1)}) + R_\alpha(p_A^{(2)}) \geq R_\alpha(w_A) \quad (0 < \alpha \leq 1), \quad (4.19)$$

$$H_\alpha(Q_{AB}^{(1)}) + H_\alpha(Q_{AB}^{(2)}) \geq H_\alpha(p_A^{(1)}) + H_\alpha(p_A^{(2)}) \geq H_\alpha(w_A) \quad (0 < \alpha < \infty). \quad (4.20)$$

For $0 < \alpha \leq 1$, the Rényi α -entropy is concave, so that the claim (4.15) follows from (4.19). The claim (4.16) follows from (4.20) due to concavity of the Tsallis α -entropy for all $\alpha > 0$. ■

In Proposition 5, we deal with separability conditions derived from majorization uncertainty relations. Unlike (4.7), the condition (4.15) is restricted to the range $0 < \alpha \leq 1$, where Rényi’s entropy is concave irrespectively to dimensionality of probabilistic vectors. For $\alpha > 1$, concavity properties actually depend on dimensionality of probabilistic vectors. So, the binary Rényi entropy is concave for all $0 < \alpha \leq 2$ [40]. Since the case of qubits is very important, we give two separability conditions additional to (4.15). Here, the majorization-based relations (3.12) and (3.15) will be used.

Since the number of outcomes $D = 2$, we consider pairs $\{\mathcal{E}_A^{(1)}, \mathcal{E}_A^{(2)}\}$ and $\{\mathcal{E}_B^{(1)}, \mathcal{E}_B^{(2)}\}$ of orthonormal bases, each in two dimensions. For product states of a two-qubit system and $\alpha > 1$, one has

$$R_\alpha(Q_{AB}^{(1)}) + R_\alpha(Q_{AB}^{(2)}) \geq \frac{2}{1 - \alpha} \ln \left(\frac{1 + \|w_S\|_\alpha^\alpha}{2} \right), \quad (4.21)$$

$$R_\alpha(Q_{AB}^{(1)}) + R_\alpha(Q_{AB}^{(2)}) \geq R_\alpha(w'_S). \quad (4.22)$$

The majorizing vectors ω_S and ω'_S are calculated for unitary 2×2 matrix $V_S = V(\mathcal{E}_S^{(1)}, \mathcal{E}_S^{(2)})$ with $S = A, B$ in line with (3.10) and (3.14). We cannot extend (4.21) and (4.22) to separable states without entropic concavity. When $d = 2$ and $1 < \alpha \leq 2$, for each separable state ρ_{AB} we finally get

$$R_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + R_\alpha(\mathcal{M}^{(2)}|\rho_{AB}) \geq \frac{2}{1-\alpha} \ln\left(\frac{1 + \|\omega_S\|_\alpha^\alpha}{2}\right), \quad (4.23)$$

$$R_\alpha(\mathcal{M}^{(1)}|\rho_{AB}) + R_\alpha(\mathcal{M}^{(2)}|\rho_{AB}) \geq R_\alpha(\omega'_S). \quad (4.24)$$

Thus, for a two-qubit system majorization-based separability conditions in terms of Rényi's entropies are obtained in the range $0 < \alpha \leq 2$. This step may extend our possibilities to detect entanglement in multi-qubit systems.

We shall now proceed to separability conditions connected with local measurements of a special structure. As such, mutually unbiased bases and symmetric informationally complete measurements can be used. In the case of MUBs, the following statement takes place.

Proposition 6 *Let $\{\mathcal{E}_A^{(1)}, \dots, \mathcal{E}_A^{(K)}\}$ be a set of K MUBs in \mathcal{H}_A , and let $\{\mathcal{E}_B^{(1)}, \dots, \mathcal{E}_B^{(K)}\}$ be a set of K MUBs in \mathcal{H}_B . Let K POVMs $\mathcal{M}^{(t)}(\mathcal{E}_A^{(t)}, \mathcal{E}_B^{(t)})$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed from these MUBs according to Definition 1. For each separable state ρ_{AB} and $\alpha \in (0; 2]$, there holds*

$$\frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{M}^{(t)}|\rho_{AB}) \geq \ln_\alpha\left(\frac{Kd}{d+K-1}\right), \quad (4.25)$$

where $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.

Proof. Again, we will prove (4.25) only for one of the cases $S = A, B$. The Tsallis α -entropy is Schur-concave for all $\alpha \in (0; 2]$. Combining this fact with (2.17) and (3.29), for any product we have

$$\frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{M}^{(t)}|\rho_A \otimes \rho_B) \geq \frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{E}_A^{(t)}|\rho_A) \geq \ln_\alpha\left(\frac{Kd}{d+K-1}\right). \quad (4.26)$$

By concavity of Tsallis' α -entropy, we extend the latter to all separable states. ■

In the particular case $\alpha = 1$, we have

$$\frac{1}{K} \sum_{t=1}^K H_1(\mathcal{M}^{(t)}|\rho_{AB}) \geq \ln\left(\frac{Kd}{d+K-1}\right), \quad (4.27)$$

whenever ρ_{AB} is separable. This separability condition is actually those that can be derived from the Rényi-entropy bound (3.28). When d is not specified, we can use concavity of Rényi's α -entropy only for $0 < \alpha \leq 1$. In addition, it does not increase with growth of α . For $d = 2$, however, the Rényi α -entropy is concave up to $\alpha = 2$. Thus, for a two-qubit system we write the condition

$$\frac{1}{K} \sum_{t=1}^K R_2(\mathcal{M}^{(t)}|\rho_{AB}) \geq \ln\left(\frac{2K}{K+1}\right), \quad (4.28)$$

where $K = 2, 3$ and ρ_{AB} is separable. The result remains formally valid for $K = 1$, but the bound becomes trivial here.

Detecting entanglement, we should use as many complementary measurement as possible. When the dimensionality d of subsystems is a prime power, $d+1$ mutually unbiased bases exist. For other values of d , we may apply mutually unbiased measurements, since a complete set of $d+1$ MUMs exists in all dimensions [51]. Hence, entanglement detection with MUMs is of interest. The following claim is derived from (3.33) similarly to the proof of Proposition 6.

Proposition 7 *Let $\{\mathcal{N}_A^{(1)}, \dots, \mathcal{N}_A^{(K)}\}$ be a set of K MUMs of the efficiency \varkappa_A in \mathcal{H}_A , and let $\{\mathcal{N}_B^{(1)}, \dots, \mathcal{N}_B^{(K)}\}$ be a set of K MUBs of the efficiency \varkappa_B in \mathcal{H}_B . Let K POVMs $\mathcal{M}^{(t)}(\mathcal{N}_A^{(t)}, \mathcal{N}_B^{(t)})$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed from these MUMs according to Definition 1. For each separable state ρ_{AB} and $\alpha \in (0; 2]$, there holds*

$$\frac{1}{K} \sum_{t=1}^K H_\alpha(\mathcal{M}^{(t)}|\rho_{AB}) \geq \ln_\alpha\left(\frac{Kd}{\varkappa_{SD}d + K - 1}\right), \quad (4.29)$$

where $S = A, B$ and $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.

Entropic uncertainty relations for symmetric informationally complete measurements also lead to separability conditions. Note that such separability conditions are formulated for a single measurement. We give formulations for a usual SIC-POVM and then for a general one.

Proposition 8 *Let \mathcal{F}_A and \mathcal{F}_B be two sets of subnormalized vectors that form SIC-POVMs in \mathcal{H}_A and \mathcal{H}_B , respectively. Let POVM $\mathcal{M}(\mathcal{F}_A, \mathcal{F}_B)$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed according to Definition 1. For each separable state ρ_{AB} and $\alpha \in (0; 2]$, there holds*

$$H_\alpha(\mathcal{M}|\rho_{AB}) \geq \ln_\alpha \left(\frac{d(d+1)}{2} \right), \quad (4.30)$$

where $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.

Proof. The Tsallis α -entropy is Schur-concave for all $\alpha \in (0; 2]$. Combining this fact with (2.17) and (3.36), for any product we have

$$H_\alpha(\mathcal{M}|\rho_A \otimes \rho_B) \geq H_\alpha(\mathcal{F}_A|\rho_A) \geq \ln_\alpha \left(\frac{d(d+1)}{2} \right). \quad (4.31)$$

By concavity of Tsallis' α -entropy, the latter is extended to all separable states. ■

General SIC-POVM exist in all finite dimensions [52]. Moreover, they can be obtained within a unifying framework. Even if usual SIC-POVMs exist in all dimensions, they may be difficult to implement. Thus, entanglement detection with general SIC-POVM may be more appropriate. The following claim is derived from (3.39) similarly to the proof of Proposition 8.

Proposition 9 *Let \mathcal{N}_A and \mathcal{N}_B be two general SIC-POVMs in \mathcal{H}_A and \mathcal{H}_B , respectively. Let POVM $\mathcal{M}(\mathcal{N}_A, \mathcal{N}_B)$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ be constructed according to Definition 1. For each separable state ρ_{AB} and $\alpha \in (0; 2]$, there holds*

$$H_\alpha(\mathcal{M}|\rho_{AB}) \geq \ln_\alpha \left(\frac{d(d+1)}{a_S d^2 + 1} \right), \quad (4.32)$$

where $S = A, B$ and $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$.

We have derived a lot of separability conditions in terms of the Rényi and Tsallis entropies. These conditions are related to the measurement scheme with projectors of the form (4.3). The violation of any of separability conditions certifies that the measured state of a bipartite system is entangled. We also note that the presented conditions actually concern biseparability. The problem of detecting multipartite entanglement is generally more complicated [68]. This issue was addressed in [69–72].

V. DISCUSSION

In this section, we will apply the presented conditions to some states, for which separability limits are already known. In particular, we compare the two types of derived criteria with each other and also with previous criteria given in the literature. Since separability of qubit systems is well studied, we begin with a two-qubit system. In the following, the corresponding Pauli matrices will be denoted as $\sigma_x, \sigma_y, \sigma_z$. Bipartite separability conditions are often tested with density matrices of the form $(1-c)\varrho_{sep} + c|\Phi\rangle\langle\Phi|$. In this formula, the density matrix ϱ_{sep} is separable, $|\Phi\rangle$ is a maximally entangled state, and real constant $c \in [0; 1]$. Taking ϱ_{sep} to be the completely mixed state, the form (5.3) is a bipartite case of Werner states [62]. A bipartite Werner state is separable if and only if [73]

$$c \leq \frac{1}{d+1}. \quad (5.1)$$

The authors of [73] also presented necessary and sufficient conditions for multipartite Werner states. For a bipartite system of two qubits, the inequality (5.1) gives $c \leq 1/3$. The entangled pure state $|\Psi\rangle$ will be taken as

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|z_0 z_0\rangle + |z_1 z_1\rangle). \quad (5.2)$$

Substituting the completely mixed states of two qubits, we consider the following one-parameter family of density matrices,

$$\varrho_\Phi = \frac{1-c}{4} \mathbb{1}_2 \otimes \mathbb{1}_2 + c|\Phi\rangle\langle\Phi|. \quad (5.3)$$

Let us take the eigenbasis $\{|z_0\rangle, |z_1\rangle\}$ of σ_z and a rotated basis $\{|u_0\rangle, |u_1\rangle\}$ such that

$$|u_0\rangle := \cos\theta |z_0\rangle + \sin\theta |z_1\rangle, \quad |u_1\rangle := \sin\theta |z_0\rangle - \cos\theta |z_1\rangle, \quad (5.4)$$

where $\theta \neq 0$. So, we wish to deal not only with two mutually unbiased bases. This point is essential for comparing majorization-based separability conditions with separability conditions of the Maassen–Uffink type. Our construction implies the choice of observables

$$\mathbf{A}^{(z)} = \mathbf{B}^{(z)} = \sigma_z, \quad \mathbf{A}^{(u)} = \mathbf{B}^{(u)} = |u_0\rangle\langle u_0| - |u_1\rangle\langle u_1|. \quad (5.5)$$

For $\theta = \pi/2$, the second basis $\{|u_0\rangle, |u_1\rangle\}$ gives the eigenbasis of σ_x , whence $\mathbf{A}^{(x)} = \mathbf{B}^{(x)} = \sigma_x$. The measurements $\mathcal{M}^{(z)}$ and $\mathcal{M}^{(u)}$ respectively contain the projectors

$$\Lambda_0^{(z)} = |z_0 z_0\rangle\langle z_0 z_0| + |z_1 z_1\rangle\langle z_1 z_1|, \quad \Lambda_1^{(z)} = |z_0 z_1\rangle\langle z_0 z_1| + |z_1 z_0\rangle\langle z_1 z_0|, \quad (5.6)$$

$$\Lambda_0^{(u)} = |u_0 u_0\rangle\langle u_0 u_0| + |u_1 u_1\rangle\langle u_1 u_1|, \quad \Lambda_1^{(u)} = |u_0 u_1\rangle\langle u_0 u_1| + |u_1 u_0\rangle\langle u_1 u_0|. \quad (5.7)$$

For θ in the quadrant I, we have $\eta_S = \max\{\cos\theta, \sin\theta\} \equiv \eta$ and

$$w = (\eta, 1 - \eta), \quad (5.8)$$

$$w' = \frac{1}{4} (1 + 2\eta + \eta^2, 3 - 2\eta - \eta^2). \quad (5.9)$$

Combining these observations with (4.7), (4.8), (4.15), (4.16), (4.23), and (4.24), we obtain a lot of separability conditions for the case considered.

When $\theta = \pi/2$, we have $\eta = 1/\sqrt{2}$ and two MUBs, namely the eigenbases of σ_z and σ_x . By calculations, we obtain

$$\langle \Phi | \Lambda_i^{(z)} | \Phi \rangle = \delta_{i0}, \quad \langle \Phi | \Lambda_j^{(x)} | \Phi \rangle = \delta_{j0}, \quad (5.10)$$

where δ_{ij} is the Kronecker symbol. On the completely mixed state, each of two measurements generates the uniform distribution. For the state (5.3), we twice obtain the pair of probabilities $(1 \pm c)/2$. By inspection, the best detection among relations of the Maassen–Uffink type is provided by (4.7) for the choice $\alpha = \infty$ and $\beta = 1/2$. Here, we have the condition

$$-\ln\left(\frac{1+c}{2}\right) + \ln\left(1 + \sqrt{1-c^2}\right) \geq \ln 2, \quad (5.11)$$

which is equivalent to $c \leq 1/\sqrt{2}$. So, the entropic separability conditions of the form (4.7) detect entanglement when $c > 1/\sqrt{2} \approx 0.7071$. The same range takes place for the criterion that considers the sum of maximal probabilities in two measurements. It is rather natural since that criterion is also based on the Maassen–Uffink approach [9]. The result quoted follows from a general formulation by substituting $d = 2$. Performing a direct optimization in the qubit case allows to improve restrictions [9]. On the other hand, this approach becomes hardly appropriate with growth of the dimensionality.

Let us proceed to majorization-based separability conditions. The condition (4.23) then gives

$$-\ln\left(\frac{1+c^2}{2}\right) \geq -\ln\left(\frac{1+\|w\|_2^2}{2}\right), \quad (5.12)$$

where $\|w\|_2^2 = 2 - \sqrt{2}$. With (5.12), we are able to detect entanglement for $c > \|w\|_2 \approx 0.7654$. Further, the condition (4.24) reads

$$-\ln\left(\frac{1+c^2}{2}\right) \geq -\ln\|w'\|_2, \quad (5.13)$$

where $\|w'\|_2$ is calculated in line with (5.9) for $\eta = 1/\sqrt{2}$. For $c > \sqrt{2\|w'\|_2 - 1} \approx 0.7450$, we can detect entanglement due to (5.13). For both the majorization-based separability conditions, the range of detection is slightly less than for conditions of the Maassen–Uffink type.

It is instructive to consider two bases that are not mutually unbiased. When $\theta = \pi/6$, the best result among conditions of the Maassen–Uffink type is reached by (4.7) for the choice $\alpha = \beta = 1$. Entropic separability conditions of the form (4.7) detect entanglement for $c > c_1$ with $c_1 \approx 0.9347$. The best result among majorization-based

conditions is provided due to (5.13). With the latter, we are able to detect entanglement for $c > c_2$ with $c_2 \approx 0.8719$. With the majorization-based separability conditions, we can reach more effective detection. For other values of θ , numerical results turned out to be qualitatively similar. Thus, separability conditions of the Maassen–Uffink type are preferable when we deal with two MUBs. Of course, mutually unbiased bases provide more appropriate schemes for entanglement detection *per se*. However, in quantum information processing we will also perform measurements designed for other purposes. Statistics of such measurements may nevertheless be used additionally for the aim of entanglement detection. Here, we should keep in mind entropic separability conditions based on majorization uncertainty relations.

Let us compare two forms of separability conditions with using three MUBs. For a qubit, these MUBs are taken as the eigenbases of the Pauli observables. The first form deals with the so-called correlation measure introduced in [57]. Using (3.30), the authors of [57] obtained separability conditions in terms of the correlation measure. In two dimensions, the correlation measure is expressed as

$$J(\rho_{AB}) = \sum_{t=z,x,y} \sum_{i=0,1} \langle e_i^{(t)} | e_i^{(t)} | \rho_{AB} | e_i^{(t)} | e_i^{(t)} \rangle, \quad (5.14)$$

where $|ee'\rangle \equiv |e\rangle \otimes |e'\rangle$. For separable states of a two-qubit system, the correlation measure satisfies $J \leq 2$. Simple calculations now give

$$J(\varrho_\Phi) = \frac{1+c}{2} + \frac{1+c}{2} + \frac{1-c}{2} = \frac{3+c}{2}. \quad (5.15)$$

For all $c \in [0; 1]$, the right-hand side of (5.15) does not violate the separability condition $J \leq 2$. Density matrices of the form (5.3) are not separable for $c > 1/3$ and all escape the entanglement detection with respect to this criterion. It is not the case for the second form of separability conditions. The latter is based on the entropic bounds (4.25) and (4.28).

For each of three measurements $\mathcal{M}^{(t)} = \{\Lambda_0^{(t)}, \Lambda_1^{(t)}\}$, where $t = z, x, y$, the projectors are written according to (5.6). By direct calculations, we have

$$\langle \Phi | \Lambda_0^{(z)} | \Phi \rangle = 1, \quad \langle \Phi | \Lambda_0^{(x)} | \Phi \rangle = 1, \quad \langle \Phi | \Lambda_1^{(y)} | \Phi \rangle = 1. \quad (5.16)$$

On the completely mixed state, each of three measurements generates the uniform distribution with two outcomes. For the state (5.3), we then obtain probabilities $(1 \pm c)/2$ in all three cases. Substituting $d = 2$, $K = 3$ and $\alpha = 2$, both the entropic bounds (4.25) and (4.28) lead to the condition

$$\left(\frac{1+c}{2}\right)^2 + \left(\frac{1-c}{2}\right)^2 = \frac{1+c^2}{2} \leq \frac{2}{3}, \quad (5.17)$$

or merely $c \leq 1/\sqrt{3}$. So, the entropic separability conditions (4.25) and (4.28) detect entanglement when $c > 1/\sqrt{3} \approx 0.5774$. In the considered example, the entropic approach has been more effective than the method using the correlation measure. An efficiency of separability conditions is very sensitive to the choice of local measurement bases. For conditions in terms of maximal probabilities, this fact was already mentioned in [13]. Also, the range $c > 1/\sqrt{3}$ is wider than the range $c > 1/\sqrt{2}$, in which separability conditions of the form (4.7) are able to detect entanglement. Our abilities to detect entanglement should increase, when the number of involved bases grows and used separability conditions are chosen properly.

Using several MUBs, we have two possible types of separability conditions respectively expressed in terms of entropies and in terms of correlation measures. It is also instructive to compare these types with entangled states of a two-qutrit system. Let us recall the generalized Pauli operators

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^* \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (5.18)$$

where $\omega_3 = \exp(i2\pi/3)$. Four MUBs in the qutrit Hilbert space can be described as the eigenbases of the operators Z , X , ZX , and ZX^2 . We shall test separability conditions on the following family of states,

$$\varrho_\Psi = \frac{1-c}{9} \mathbb{1}_3 \otimes \mathbb{1}_3 + c |\Psi\rangle\langle\Psi|, \quad (5.19)$$

$$|\Psi\rangle = \frac{1}{\sqrt{3}} (|z_0x_0\rangle + |z_1x_2\rangle + |z_2x_1\rangle). \quad (5.20)$$

By $\{|z_i\rangle\}$, $\{|x_i\rangle\}$, $\{|y_i\rangle\}$, where $i = 0, 1, 2$, we mean the eigenbases of Z , X , ZX , respectively. Defining the correlation measure for three MUBs similarly to (5.14), calculations give $J(\rho_\Psi) = (1 + 2c)/3$. Separability conditions in terms of correlation measures follow from (3.30). When $d = 3$ and $K = 3$, for all separable states we have the condition $J \leq 5/3$. The latter is satisfied by the calculated $J(\rho_\Psi)$ for all $c \in [0; 1]$. Density matrices of the form (5.19) are not separable for $c > 1/4$ and all escape the entanglement detection with respect to this criterion. Note that the pure state (5.20) is an eigenstate of three operators, namely

$$(Z \otimes X)|\Psi\rangle = |\Psi\rangle, \quad (X \otimes Z)|\Psi\rangle = |\Psi\rangle, \quad (ZX \otimes ZX)|\Psi\rangle = \omega_3|\Psi\rangle. \quad (5.21)$$

Hence, we may try to rotate unitarily local measurement bases. For instance, one can take simultaneously $\{|z_i\rangle\}$ on qutrit A and $\{|x_i\rangle\}$ on qutrit B , and so on. So, we calculate the measure

$$\sum_{i=0,1,2} \left(\langle z_i x_i | \rho_\Psi | z_i x_i \rangle + \langle x_i z_i | \rho_\Psi | x_i z_i \rangle + \langle y_i y_i | \rho_\Psi | y_i y_i \rangle \right) = \frac{1 + 2c}{3}, \quad (5.22)$$

whose left-hand side is hinted by (5.21). However, we still see no violation of separability conditions. To get detection, we should correspondingly permute kets in local bases of one of two qutrits.

Let us proceed to three measurements designed according to Definition 1. To the first pair of bases, we assign the three projectors

$$\Pi_0^{(zx)} = |z_0 x_0\rangle\langle z_0 x_0| + |z_1 x_2\rangle\langle z_1 x_2| + |z_2 x_1\rangle\langle z_2 x_1|, \quad (5.23)$$

$$\Pi_1^{(zx)} = |z_0 x_1\rangle\langle z_0 x_1| + |z_1 x_0\rangle\langle z_1 x_0| + |z_2 x_2\rangle\langle z_2 x_2|, \quad (5.24)$$

$$\Pi_2^{(zx)} = |z_0 x_2\rangle\langle z_0 x_2| + |z_1 x_1\rangle\langle z_1 x_1| + |z_2 x_0\rangle\langle z_2 x_0|, \quad (5.25)$$

which form $\mathcal{M}^{(zx)}$. In a similar manner, we write projectors of the measurements $\mathcal{M}^{(xz)} = \{\Pi_k^{(xz)}\}$ and $\mathcal{M}^{(yy)} = \{\Pi_k^{(yy)}\}$. It immediately follows that

$$\langle \Psi | \Lambda_0^{(zx)} | \Psi \rangle = 1, \quad \langle \Psi | \Lambda_0^{(xz)} | \Psi \rangle = 1, \quad \langle \Psi | \Lambda_1^{(yy)} | \Psi \rangle = 1. \quad (5.26)$$

On the completely mixed state, each of three measurements generates the uniform distribution with three outcomes. For the state (5.19), we obtain three probability distributions, each with one entry $(1 + 2c)/3$ and two entries $(1 - c)/3$. Substituting $d = 3$, $K = 3$ and $\alpha = 2$, the entropic bound (4.25) and (4.28) gives the condition

$$\left(\frac{1 + 2c}{3} \right)^2 + 2 \left(\frac{1 - c}{3} \right)^2 = \frac{1 + 2c^2}{2} \leq \frac{5}{9}, \quad (5.27)$$

or $c \leq 1/\sqrt{3}$. So, the entropic separability condition (4.25) can detect entanglement of (5.19) when $c > 1/\sqrt{3} \approx 0.5774$. We again see cases when separability conditions of the entropic type are more efficient than separability conditions in terms of correlation measures. Of course, both the types essentially depend on local unitary rotations and permutations of kets in bases. However, such operations will considerably increase costs of entanglement detection. In practice, when resources are fixed, we should therefore try to use as many separability conditions as possible.

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